

On the $1/D$ expansion for directed polymers

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Abstract. We present a variational approach for directed polymers in D transversal dimensions which is used to compute the correction to the mean field theory predictions with broken replica symmetry. The trial function is taken to be a symmetrized version of the mean-field solution, which is known to be exact for $D = \infty$. We compute the free energy corresponding to that function and show that the finite- D corrections behave like $D^{-4/3}$. It means that the expansion in powers of $1/D$ should be used with great care here. We hope that the techniques developed in this note will be useful also in the study of spin glasses.

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1 Introduction

At the present moment a very useful approach in the study of phase transitions in disordered systems is given by the mean-field theory in the framework of the spontaneous breaking of the replica symmetry [1]. In many cases there is a well-developed theoretical understanding of the mean-field theory, using the replica formalism or using the equivalent probabilistic cavity approach. The situation is not so good if we consider the perturbative corrections to the mean field approach in the phase where replica symmetry is broken: a few computations exist, but they are very difficult from the technical point of view, so that only one loop contributions have been computed [2,3].

The aim of this paper is to explore a new field: non perturbative effects. Only very few papers exist which are devoted to the study of non perturbative effects, among them we recall [4–7]. Non perturbative effects are very important because they could spoil the predictions of mean field theory in a rather subtle way, which cannot be detected by the usual perturbative methods. In this note we will concentrate our attention on the following problem. In the phase where replica symmetry is broken, in replica space there are many different stationary points for the free energy as a function of the order parameter (let us call it $Q_{a,b}$ and denote its average in a given stationary point by $q_{a,b}$). These stationary points are related to the other by permutations. In the usual approach one pick up a particular solution and the symmetrization is done explicitly when one computes the observables. For example

one writes

$$\langle Q_{a,b}^2 \rangle = \frac{1}{n!} \sum_{\Pi} q_{\Pi(a),\Pi(b)}^2, \quad (1)$$

where Π denotes a permutation, $\Pi(a)$ is value of the index a after the permutation and the sum is done over all the $n!$ permutations of n elements ($n \rightarrow 0$ at the end).

This symmetrization is crucial when we need to compute quantities in finite volume because in this situation replica symmetry must be restored. The physical interpretation of spontaneously broken replica symmetry as the coexistence of infinite many pure states is deeply rooted on this procedure. For example in the case of spin glasses we have that

$$\frac{1}{N^p} \sum_{i_1, i_2, \dots, i_p=1, N} \langle \sigma_{i_1} \sigma_{i_2} \dots \sigma_{i_p} \rangle^2 \approx \langle Q_{a,b}^p \rangle = \frac{1}{n!} \sum_{\Pi} q_{\Pi(a),\Pi(b)}^2. \quad (2)$$

Things however become more complicated in systems which have one direction much longer than the others. Let us consider a system with NL degrees of freedom (*e.g.* $\sigma_{k,t}$), where $k = 1, \dots, N, t = 1, \dots, L$) and let us suppose that the interaction has a finite range in the variable t . We are interested in the case where L is very large (infinite) and N is large, but definitely much smaller than L . General arguments tell us that the correlation length in the t direction must be finite as soon N is finite: the system becomes one-dimensional and in this situation no spontaneous breaking of a symmetry is possible.

For large N we can formally associate an order parameter to each system at fixed t . Systems at quite

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different t will have order parameters which point into different directions in the order parameter space (in our case replica space); indeed standard arguments imply that also if we constraint the order parameter at time zero to point in a given direction the symmetry will be restored at large times. This restoration is related to tunneling events among configurations in which the order parameters have different orientations in replica space. In this situation the zero modes connected to the spontaneous breaking of the replica symmetry are lifted and a finite correlation length (in the t direction) appears in the problem. Of course this correlation length diverges in the limit when N goes to infinity.

This kind of problems appears in many cases, for example if we study the dynamics of spin glasses and we are interested in the tunneling among different pure states. A similar problem arises also in the equilibrium behavior of samples that are much longer in one direction than the other.

In this paper we study these problems in a different context, the equilibrium behavior of infinitely long directed polymer. In the replica approach one finds that at low temperature the replica symmetry is broken. This breaking is an artefact of the mean field approximation and replica symmetry is eventually restored in this problem. In this case it is crucial to find out the solution to the problem of tunneling among different ground states, which is likely to control the large time behavior of the correlations. The number of degrees of freedom of a directed polymer embedded in a D -dimensional space is proportional to D . The replica approach can be successfully used when D goes to infinity, but for fixed D we have to deal with the existence of tunneling effects and replica symmetry restoration at large times. It is quite possible that these effects are responsible for the inaccurate value of the exponent computed at low dimensions in the replica approach.

In this note we present a tentative computation of tunneling effects in the case of breaking of the replica symmetry in a continuous way, *i.e.* the function $q(u)$ is continuous¹. We will show that one can define infinitesimal permutations and the tunneling among configurations differing by an infinitesimal permutation is not exponentially suppressed and gives a contribution that decreases as a power of D at large dimensions.

In the general case the computations are rather involved. In the present case the analysis is simple because we can have to find the ground state of the Schroedinger equation for n interacting particles. The effects of the tunneling among wave functions with different order parameter may be taken into account in a first approximation by using a symmetrized wave function and computing the effects due to the superposition of different wave functions.

This paper is organized as follows: in Section 2 we recall some of the results of the replica approach to directed polymer, we present in the next section the symmetrized

trial wavefunction and we compute the corresponding free energy. Section 4 is devoted to the discussion of the integral over permutations and its consequences to the finite- D corrections. The last section summarizes the results obtained.

2 Mean-field solution

The directed polymer in a random external field is described by the Hamiltonian [9–11]

$$h[\omega] = \int dt \left(\frac{1}{2} \left(\frac{\partial \omega}{\partial t} \right)^2 + V(t, \omega(t)) \right) \quad (3)$$

where t is the coordinate along the polymer and the D -component vector ω describes the transversal coordinate. The random external field is supposed to be Gaussian-distributed with the correlation

$$\overline{V(t, \omega)V(t', \omega')} = -D\delta(t - t')f\left(\frac{|\omega - \omega'|}{\sqrt{D}}\right). \quad (4)$$

The relevant part of the correlation is the long-distance behavior, which we choose to be power-law

$$f(x) \simeq \frac{g}{2(1-\gamma)}x^{2(1-\gamma)}, \quad x \rightarrow \infty. \quad (5)$$

In this approach the directed polymer is a classic-mechanical static object, so that the interesting properties are related to its shape. Moreover, we neglect the size of the particular monomers. In this way there is no intrinsic natural length scale in the system.

The scaling exponents are crucial to describe the large distance behaviour of the system. In particular ζ controls the transversal fluctuations

$$\overline{(\omega(t) - \omega(t'))^2} \sim (t - t')^{2\zeta}. \quad (6)$$

Sometimes ζ is called wandering exponent.

The exponent χ controls the sample-to-sample fluctuations of the free energy of a finite part of the polymer, of the length L

$$\overline{F(L)^2} - \overline{F(L)}^2 \sim L^{2\chi}. \quad (7)$$

There is a scaling formula relating ζ and χ , resulting from the Galilean invariance: $\chi = 2\zeta - 1$ [12, 13].

Different approaches to the calculation of the scaling exponents are possible [11, 13–15], among them the replica method is one of the most powerful. Making use of the replica trick, the problem is equivalent to the system of n particles, where n is the number of replicas, with pairwise interaction determined by the correlation function of the random external field. The exact solution is known for two opposite cases. For $D = 1$ and δ -correlated disorder the Bethe *Ansatz* leads to the value $\zeta = \frac{2}{3}$.

On the other hand, simple scaling argument (see *e.g.* [16]) leads to the Flory result

$$\zeta = \zeta_F \equiv \frac{3}{2(1+\gamma)}. \quad (8)$$

¹ In a separate publication we will discuss in more details the simpler case where one the replica symmetry is broken at one step, *i.e.* the function $q(u)$ takes only two values [8].

The same scaling analysis gives us the relation for the effective value of γ for the case of δ -correlated disorder $\gamma = 1 + D/2$, thus the Flory result does not coincide with the exact solution for $D = 1$. The question of the validity of the Flory formula arises and it is partially answered by the results obtained in another case, where the exact solution is known, namely in the $D = \infty$ limit [11]. In this case, the mean field approach should give the correct result.

The mean-field equations can be obtained in a number of ways. In the most transparent method one consider the perturbation expansion for the propagator $G_{ab}(t - t') = \langle \omega_a(t) \omega_b(t') \rangle$, where a and b are the replica indices. The role of the order parameter is played by the self-energy σ_{ab} defined by

$$G_{ab}(t) = \left(\left[\frac{\partial^2}{\partial t^2} - \sigma \right]^{-1} \right)_{ab}. \quad (9)$$

The point is that the Hartree-Fock approximation is exact in the $D = \infty$ limit (D corresponds to the number of field components), thus giving the desired mean-field equations for σ . Only cactus diagrams survive in this limit, thus reproducing the Hartree-Fock expansion. Non-cactus diagrams give formally corrections which are proportional to $1/D$. Two cases are possible:

- In the situation called ‘short range correlations’, the matrix σ is either replica-symmetric or it corresponds to one step of replica-symmetry breaking. This solution is thermodynamically stable for $\gamma > 2$. The resulting wandering exponent is $\zeta = \frac{1}{2}$, *i.e.* no influence of the disorder is seen.
- For $\gamma < 2$ another solution should be looked for and the natural choice is to break the replica symmetry in the hierarchical manner already familiar from the spin-glass theory [1]. The matrix σ_{ab} is then parametrized by the pair $(\tilde{\sigma}, \sigma(u))$, where $\sigma(u)$ is a function on the interval $(0,1)$ and the number $\tilde{\sigma}$ is the diagonal element of the matrix σ_{ab} . At the end we arrive at a surprising result stating that the wandering exponent has exactly the Flory value.

The problem can be formulated in an alternative way by using the Feynman-Katz representation for the solution of the Schroedinger equation. In this case, the usual replica approach is equivalent to finding the ground state of the imaginary-time quantum Hamiltonian. If we use the variational principle and we suppose that the trial wavefunction $(|\psi\rangle)$ is Gaussian, *i.e.*

$$\langle \omega | \psi \rangle = \exp(-Q_{ab}^{-1} \omega_a \omega_b), \quad (10)$$

we recover the results of the previously described replica approach. This approximation corresponds to taking into account only one of the minima of the free energy and supposing that this valley has parabolic shape, *i.e.* it corresponds to the D -dimensional harmonic oscillator, whose ground state wave function is Gaussian. The matrix Q^{-1} of variational parameters is then equivalent to the self-energy in the Hartree-Fock approach.

3 Symmetrized wavefunction

3.1 General considerations

In case where the transverse spatial dimensionality D is finite, the trial wavefunction should be generalized in such a way that all the equivalent valleys are taken into account. The situation is analog to the double-well potential where the barrier between the valleys are proportional to D . For D finite we have to include tunneling from one valley to the other one. The relation between the equivalent valleys is provided by the symmetry of the problem, in our case by the permutations of the n replicas. We denote S_n the group of all such permutations and S_n^1 the subset of S_n obtained by excluding the identity.

The problem we want to solve is to find (in the limit $n \rightarrow 0$) the ground state of the Hamiltonian

$$H = -\frac{1}{2} \sum_{a=1}^n \left(\frac{\partial}{\partial \omega_a} \right)^2 + D \sum_{a \neq b} f \left(\frac{1}{\sqrt{D}} |\omega_a - \omega_b| \right), \quad (11)$$

where the attractive inter-particle potential can have two forms

$$f(x) = -g\delta(x) \quad (12)$$

or

$$f(x) = \frac{g}{2(1-\gamma)} x^{2(1-\gamma)}. \quad (13)$$

The scaling analysis suggests the first case corresponds to the second case for $\gamma = 1 + D/2$. We will use the form (13) hereafter.

In the usual approach one minimize the free energy functional

$$F = \frac{1}{nD} \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} \quad (14)$$

in the space of Gaussian wave functions. Here we want to generalize this approach by minimizing the same functional in the space of the symmetrized wavefunctions of the Gaussian type

$$|\psi\rangle = \sum_{\Pi \in S_n} \Pi |\psi_1\rangle \quad (15)$$

$$\langle \omega | \psi_1 \rangle = \exp \left(-\frac{1}{4} \sum_{a,b} Q_{ab}^{-1} \omega_a \cdot \omega_b \right) \quad (16)$$

with the matrix Q as a variational parameter. This choice corresponds to the prescription used in [11], where the role of Q^{-1} plays the self-energy, which is constant in longitudinal momentum space. The permutation Π of the replicas acts as a transformation of the matrices Q and Q^{-1} :

$$Q_{ab} \rightarrow \Pi Q_{ab} \equiv Q_{\Pi(a)\Pi(b)}, \quad Q_{ab}^{-1} \rightarrow \Pi Q_{ab}^{-1} \equiv Q_{\Pi(a)\Pi(b)}^{-1}. \quad (17)$$

Unfortunately enough we have not been able to write in a closed form the sum over the permutation in the case where the matrix Q is such to break explicitly the replica symmetry. Therefore we must resort to an approximate evaluation of the quantity defined in equation (15).

3.2 Infinitesimal permutations

Let us denote $\delta Q^{-1} = \frac{1}{2}(\Pi Q^{-1} - Q^{-1})$ the difference between the permuted and the original matrix. Naively speaking we expect that for most of the permutations, for which δQ^{-1} is a quantity of order 1, the contribution to equation (15) is very small for large D , because the overlap between the original and permuted wavefunction tends to zero as the transversal dimensionality D goes to infinity. This is certainly true as long as the number of replicas remains greater than one.

There are technical difficulties arise when we perform the replica limit $n \rightarrow 0$. One of the peculiar things happening here is that the Schwartz inequality is no more valid (the space of the Q has negative dimensionality when $n < 1$) and in many cases it gets reversed. Sometimes one finds that overlap of the original and permuted vector blows up instead of vanishing when $D \rightarrow \infty$. However, we can relate this problem to the set of strange phenomena in the replica method such as the change of the minima of the free energy into maxima which arise as an effect of the analytic continuation in n . The common procedure is to take the $n \rightarrow 0$ limit after all other calculations, which involves some changes of order of limits. In this case we will make the $D \rightarrow \infty$ limit first and the replica limit ($n \rightarrow 0$) after that. More specifically, we will compute the integrals of the type $\int dx \exp(\alpha(n)Df(x))$ by saddle point method in the region $n > 1$, where α is negative and then we will continue the result to $n < 1$, where α is positive and strictly speaking the integral is not defined. However, in fact, the integral represents the sum over negative number of terms, thus the ‘undefined’ integral should be actually defined by the procedure we use to compute it or by another methodologically similar procedure.

If we exclude large permutations, we expect that only the infinitesimal permutations, *i.e.* those for which δQ^{-1} is small, will contribute. These infinitesimal permutation are not present for integer n and they arise only as an effect of the analytic continuation in n and of the peculiar way of breaking the replica symmetry.

The infinitesimal permutations we will sum over were already classified by Goltsev [17]. In his classification the exchanges of two replicas play crucial role as generators of the whole group of permutations. Clearly, the pairwise exchanges are not the only infinitesimal permutations; one should include the permutations of three, four *etc.* replicas as well, because we do not refer to the set of generators but to the set of permutations with small δQ^{-1} in general. Nevertheless, we will use the pairwise exchanges of *blocks* of replicas as typical representatives of the whole set of infinitesimal permutations with the hope that they cover that set densely enough to give reasonable results.

We suppose that the matrix Q^{-1} has the usual hierarchical form [18]

$$\begin{aligned} Q_{aa}^{-1} &= \tilde{q} = q(1) \\ Q_{ab}^{-1} &= q(x), \quad x \in (0, 1) \end{aligned} \quad (18)$$

where $x = a \cap b$ is the overlap (ultrametric co-distance) between the different replicas a and b . Thus even the matrix

Q , which is the inverse of Q^{-1} , has the same hierarchical structure

$$\begin{aligned} Q_{aa} &= \tilde{r} = r(1) \\ Q_{ab} &= r(x), \quad x \in (0, 1) \end{aligned} \quad (19)$$

and explicit relations between the pairs $(\tilde{q}, q(x))$ and $(\tilde{r}, r(x))$ can be found in the Appendix of [11]. We will frequently use these relations in the last part of the calculations. We use double notation for the value of the diagonal element for convenience in the following calculations.

Let us now define the permutations we will take into account. We will take two blocks of replicas of the size m_B and of the mutual co-distance m . Clearly $m < m_B$. The permutation in question, which we will denote $\Pi_2(m_B, m)$ will involve the exchange of these blocks as fixed units. We will use the notation $S_n^{(2)}$ for the set of all permutations of this type, the identity excluded.

Denoting the chosen blocks by the indices α and β , $\alpha \neq \beta$ and the single replicas within these blocks by pairs (α, i) and (β, j) it means that the following relations hold for the overlaps

$$\begin{aligned} (\alpha, i) \cap (\beta, j) &= m \quad \forall i, j \\ (\alpha, i) \cap (\alpha, j) &= (\beta, i) \cap (\beta, j) = m_B \quad \forall i, j. \end{aligned} \quad (20)$$

The explicit expression for the charge in the matrix Q^{-1} is

$$\begin{aligned} \delta Q_{(ai)(bj)}^{-1} &= \frac{1}{2} \left(\delta_{a\alpha} \bar{\delta}_{b\alpha} \bar{\delta}_{b\beta} \left(Q_{(\beta i)(bj)}^{-1} - Q_{(\alpha i)(bj)}^{-1} \right) \right. \\ &\quad + \delta_{a\beta} \bar{\delta}_{b\alpha} \bar{\delta}_{b\beta} \left(Q_{(\alpha i)(bj)}^{-1} - Q_{(\beta i)(bj)}^{-1} \right) \\ &\quad + \delta_{b\alpha} \bar{\delta}_{a\alpha} \bar{\delta}_{a\beta} \left(Q_{(ai)(\beta j)}^{-1} - Q_{(ai)(\alpha j)}^{-1} \right) \\ &\quad \left. + \delta_{b\beta} \bar{\delta}_{a\alpha} \bar{\delta}_{a\beta} \left(Q_{(ai)(\alpha j)}^{-1} - Q_{(ai)(\beta j)}^{-1} \right) \right). \end{aligned} \quad (21)$$

We will rewrite the expression for the free energy

$$F = \frac{1}{nD} \frac{\langle \psi_1 | H | \psi_1 \rangle + \sum_{\Pi \in S_n^1} \langle \psi_1 | H | \psi_\Pi \rangle}{\langle \psi_1 | \psi_1 \rangle + \sum_{\Pi \in S_n^1} \langle \psi_1 | \psi_\Pi \rangle} \quad (22)$$

where $|\psi_\Pi\rangle = \Pi|\psi_1\rangle$ and $|\psi_1\rangle$ was defined in (16). We now face the problem of evaluating the various terms in the previous equation as function of the infinitesimal permutation and eventually to sum over all the infinitesimal permutations.

3.3 Some explicit formulae

Here we evaluate the various terms which appear in equation (22) and we postpone the sum over the infinitesimal perturbation to the next section.

For the overlap between the original and the perturbed wavefunction we find

$$\begin{aligned} \langle \psi_1 | \psi_\Pi \rangle &= \det^{-D/2} (Q^{-1} (1 + Q \delta Q^{-1})) \\ &= \langle \psi_1 | \psi_1 \rangle \exp \left(-\frac{D}{2} T_\Pi \right). \end{aligned} \quad (23)$$

$$\begin{aligned} \langle \psi_{II} | D \sum_{a \neq b} f \left(\frac{1}{\sqrt{D}} |\omega_a - \omega_b| \right) | \psi_1 \rangle &= \sum_{a \neq b} \frac{1}{2} \hat{g} D \left\{ \frac{1}{1-\gamma} (Q_{aa} + Q_{bb} - 2Q_{ab})^{1-\gamma} - (Q_{aa} + Q_{bb} - 2Q_{ab})^{-\gamma} \right. \\ &\quad \times ([Q\delta Q^{-1}Q]_{aa} + [Q\delta Q^{-1}Q]_{bb} - 2[Q\delta Q^{-1}Q]_{ab}) + (Q_{aa} + Q_{bb} - 2Q_{ab})^{-\gamma} \\ &\quad \left. \times ([Q(\delta Q^{-1}Q)^2]_{aa} + [Q(\delta Q^{-1}Q)^2]_{bb} - 2[Q(\delta Q^{-1}Q)^2]_{ab}) + \dots \right\} \langle \psi_{II} | \psi_1 \rangle. \end{aligned} \quad (26)$$

$$\begin{aligned} -\frac{1}{2} \text{Tr}(Q\delta Q^{-1})^2 &= - \left(m_B \int_m^{m_B} dx (r(x) - r(m)) \right) (q(x) - q(m))^2 + m_B \left(\tilde{r} - \int_{m_B}^1 r(y) dy - m_B r(m) \right) \\ &\quad \times \left[\int_m^{m_B} dx \left(\tilde{r} - \int_x^1 r(y) dy - xr(x) \right) (q(x) - q(m))^2 - \int_m^{m_B} dx \int_m^{m_B} dy (r(\min(x, y)) - r(m))(q(x) - q(m))(q(y) - q(m)) \right]. \end{aligned} \quad (29)$$

where the function T_{II} can be expanded in powers of δQ^{-1} .

$$T_{II} = \text{Tr}Q\delta Q^{-1} - \frac{1}{2} \text{Tr}Q\delta Q^{-1}Q\delta Q^{-1} + \dots \quad (24)$$

This power expansion will be the starting point of the following computations. Of course, the power of δQ^{-1} does not tell us what is the order of smallness of the certain term, because the magnitude of δQ^{-1} itself depends on the parameters of the permutation in question. Thus, we will proceed by choosing an infinitesimal permutation and then collecting the terms of the same order of magnitude, which will arise during the computation of the value of the subsequent terms in (24).

Just the same consideration holds for the matrix elements of the Hamiltonian between the original and permuted state. The kinetic part of the Hamiltonian gives

$$\begin{aligned} \langle \psi_{II} | -\frac{1}{2} \sum_a \Delta_a | \psi_1 \rangle &= \frac{D}{2} \left[\text{Tr}Q^{-1} + \frac{1}{2} \text{Tr}\delta Q^{-1}Q\delta Q^{-1} \right. \\ &\quad \left. - \frac{1}{2} \text{Tr}\delta Q^{-1}Q\delta Q^{-1}Q\delta Q^{-1} + \dots \right] \langle \psi_{II} | \psi_1 \rangle \end{aligned} \quad (25)$$

and the interaction term

see equation (26) above.

The first terms in each of these expansions correspond to the mean field ($D = \infty$) approach [19].

$$H_0 = F_{D=\infty} = \frac{1}{2} \tilde{q} + \frac{1}{2} \hat{g} \frac{1}{1-\gamma} \int_0^1 dx (2(\tilde{r} - r(x)))^{1-\gamma}. \quad (27)$$

As we will see in the following, the lowest order of magnitude in the smallness of the permutation corresponds to the first and second order in δQ^{-1} , *i.e.* the terms up to the order $O((\delta Q^{-1})^2)$ are necessary.

The rescaled interaction constant \hat{g} arises in the calculation. It depends on the bare coupling g and the transverse dimensionality D . For $D \rightarrow \infty$ the two coupling constants coincide, $\hat{g} \rightarrow g$ [11]. Because the interesting effects do not regard the rescaling of the interaction constant, we chose \hat{g} independent of D and g having the corresponding D -dependence.

When computing the replica summations involved in (25, 26) we use the tree diagrams which describe the configuration of replicas imposed by the ultra-metric structure of the replica space. Let us turn to the value of T_{II} first. The lowest order in δQ^{-1} corresponds to two non-zero trees shown in Figure 1a. The corresponding value is then

$$\text{Tr}Q\delta Q^{-1} = 2m_B \int_m^{m_B} dx (r(x) - r(m))(q(x) - q(m)). \quad (28)$$

Similarly, the trees contributing to the following term in the expansion of T_{II} are depicted in Figures 1b, 1c and the value is

see equation (29) above.

We will suppose that the function $q(x)$ has the same form as the one found in the case $D = \infty$, *i.e.*

$$\begin{aligned} q(x) &= Ax^s \quad x \in (0, x_c) \\ q(x) &= Ax_c^s \quad x \in (x_c, 1) \\ \tilde{q} &= \mu + \int_0^1 q(x) dx. \end{aligned} \quad (30)$$

The parameter μ was introduced in order to have all quantities finite. It corresponds to enclosing the system to a finite volume of the linear dimension $1/\mu$. Using the relations between $q(x)$ and $r(x)$ listed in [11] we find for $m_B < x_c$

$$\begin{aligned} T_{II} = T(m_B, m) &= -\frac{1}{3}(s+1)^2 \left(1 - \frac{m}{m_B} \right)^3 \\ &\quad + O \left(\left(1 - \frac{m}{m_B} \right)^4 \right) \end{aligned} \quad (31)$$

while for $m_B > x_c$

$$\begin{aligned} T_{II} &= -\frac{m_B}{x_c} \left(\frac{m_B}{x_c} - 1 \right) \left(1 - \frac{m}{x_c} \right)^2 \\ &\quad + O \left(\left(1 - \frac{m}{x_c} \right)^3 \right). \end{aligned} \quad (32)$$

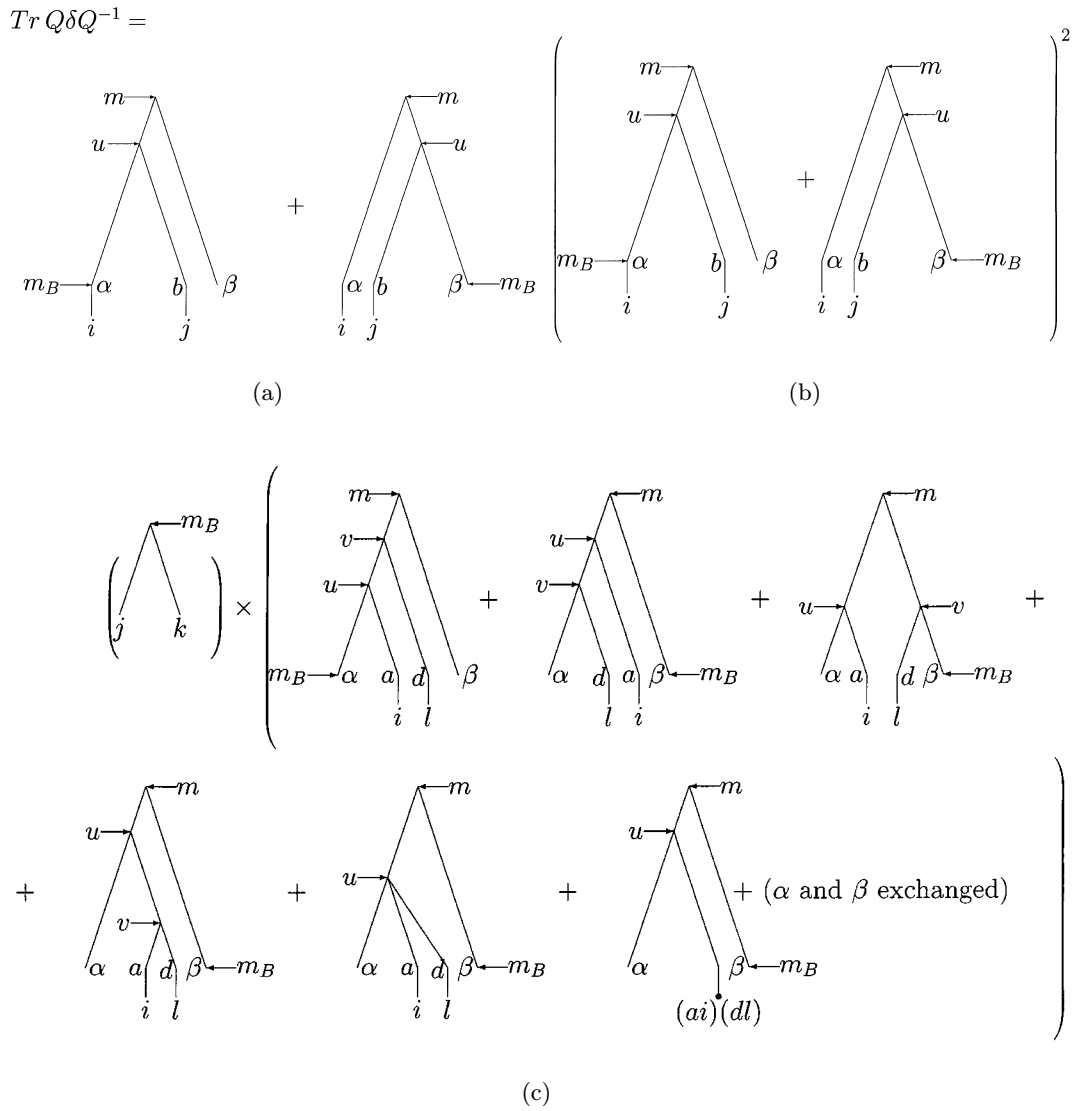


Fig. 1. Tree diagrams corresponding to the summations over replica indices in the equation (24).

Hence, we see that the values of m which dominate the sum over the permutations are those near (and smaller than) m_B for $m_B < x_c$ and those near x_c for $m_B > x_c$. If we suppose the matrix elements of the Hamiltonian to have the same power dependence on $1 - m/m_B$ and on $1 - m/x_c$ in the respective regions of m_B , the dominating D -dependence of the free energy comes from the permutations with $m_B < x_c$. In the following we will see that

the matrix elements of the Hamiltonian actually do have this property. Thus, the sum over the permutations means an integral over the parameters $m, m_B, m_B \in (0, x_c)$ and $m \in (0, m_B)$.

Similarly we compute the terms arising in $\langle \psi_{II} | H | \psi_1 \rangle$. Summing of all tree diagrams gives for the kinetic part

see equation (33) below.

$$\begin{aligned}
 \frac{1}{4} Tr(\delta Q^{-1})^2 Q \equiv H_1(m_B, m) &= \frac{m_B}{4} (\bar{r} - \int_{m_B}^1 r(y) dy - m_B r(m)) \int_m^{m_B} dx (q(x) - q(m))^2 \\
 &+ \frac{m_B}{4} \left[\int_m^{m_B} dx \left(\int_x^1 r(y) dy + x r(x) - \bar{r} \right) (q(x) - q(m))^2 + \int_m^{m_B} dx \int_m^{m_B} dy (r(\min(x, y)) \right. \\
 &\left. - r(m)) (q(x) - q(m)) (q(y) - q(m)) \right].
 \end{aligned} \tag{33}$$

$$\begin{aligned}
H_2(m_B, m) &= \frac{1}{2} \hat{g} \sum_{a \neq b} (Q_{aa} + Q_{bb} - 2Q_{ab})^{-\gamma} ([Q\delta Q^{-1}Q]_{aa} + [Q\delta Q^{-1}Q]_{bb} - 2[Q\delta Q^{-1}Q]_{ab}) \\
&= \hat{g} 2^{1-\gamma} \left\{ \int_m^1 dx \int_m^1 dy [(\Theta(x, x) - \Theta(x, y))(\tilde{r} - r(\min(x, y)))^{-\gamma} + (\Theta(x, x) + \Theta(x, y))(\tilde{r} - r(m))^{-\gamma}] \right. \\
&\quad + \int_m^1 dx [(2\Theta(x, 1) - \Theta(x, x) - \Theta(1, 1))(\tilde{r} - r(x))^{-\gamma} - (2\Theta(x, 1) + \Theta(x, x) - \Theta(1, 1))(\tilde{r} - r(m))^{-\gamma}] \\
&\quad \left. + 2\Theta(1, 1)(\tilde{r} - r(m))^{-\gamma} \right\}, \tag{35}
\end{aligned}$$

$$\begin{aligned}
\Theta(x, y) &= \frac{m_B}{2} (\Phi_0(x, y) + \Phi_0(y, x) + \Phi_1(x, y) + \Phi_1(y, x)) \\
\Phi_0(x, y) &= (r(x) - r(m)) \int_m^{m_B} dz (q(x) - q(m))(r(\min(z, y)) - r(m)) \\
\Phi_1(x, y) &= (r(x) - r(m))(q(y) - q(m))(yr(y) + \int_y^1 dz r(z) - \tilde{r}) \text{ for } y \in (m, m_B) \\
&= 0 \text{ otherwise.} \tag{36}
\end{aligned}$$

$$\begin{aligned}
H_3(m_B, m) &= -\frac{1}{2} \hat{g} \sum_{a \neq b} (Q_{aa} + Q_{bb} - 2Q_{ab})^{-\gamma} ([Q(\delta Q^{-1}Q)^2]_{aa} + [Q(\delta Q^{-1}Q)^2]_{bb} - 2[Q(\delta Q^{-1}Q)^2]_{ab}) \\
&= -\hat{g} 2^{1-\gamma} \frac{m_B^2}{2} \left\{ \int_m^1 dx \int_m^1 dy [(r(x) - r(m))^2 - (r(x) - r(m))(r(y) - r(m))](\tilde{r} - r(\min(x, y)))^{-\gamma} \right. \\
&\quad + ((r(x) - r(m))^2 + (r(x) - r(m))(r(y) - r(m)))(\tilde{r} - r(m))^{-\gamma}] + 2(\tilde{r} - r(m))^{2-\gamma} \\
&\quad + \int_m^1 dx [(\tilde{r} - r(x))^{-\gamma} (2(r(x) - r(m))(\tilde{r} - r(m)) - (r(x) - r(m))^2 - (\tilde{r} - r(m))^2) \\
&\quad \left. - (\tilde{r} - r(m))^{-\gamma} (2(r(x) - r(m))(\tilde{r} - r(m)) + (r(x) - r(m))^2 - (\tilde{r} - r(m))^2) \right\} \\
&\quad \times \int_m^{m_B} dx (xr(x) + \int_x^1 dy r(y) - \tilde{r})(q(x) - q(m))^2 + O((m_B - m)^4). \tag{37}
\end{aligned}$$

$$\begin{aligned}
H_2(m_B, m) &= \hat{g} 2^{1-\gamma} \left(-\frac{(s+1)^2}{As(s+2)} \right)^{-\gamma} \left\{ 2 \frac{(s+1)^2}{As(s+2)} \left[\left(\frac{1}{m_B^{s+2}} - \frac{1}{x_c^{s+2}} + \frac{s+2}{s+1} \frac{1}{x_c^{s+1}} - m_B \left(\frac{1}{m_B^{s+2}} \left(1 - \frac{s+2}{s+1} m_B \right) \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{1}{x_c^{s+2}} \left(1 - \frac{s+2}{s+1} x_c \right) \right) \right] + \frac{s+1}{As} (1 - \gamma + m_B) \frac{1}{m_B^{s+1}} \right\} \left(\frac{1}{m_B^{s+2}} - \frac{1}{x_c^{s+2}} + \frac{s+2}{s+1} \frac{1}{x_c^{s+1}} \right)^{-\gamma} \\
&\quad \times \frac{1}{3} (s+1)^2 \left(1 - \frac{m}{m_B} \right)^3 + O((1 - m/m_B)^4). \tag{38}
\end{aligned}$$

$$\begin{aligned}
H_3(m_B, m) &= \\
&- \hat{g} 2^{1-\gamma} \frac{m_B^2}{2} \left\{ \int_{m_B}^1 dx \left[\phi^2(x) \int_{m_B}^x dy \tilde{\phi}^{-\gamma}(y) + (1-x)\phi^2(x)\tilde{\phi}^{-\gamma}(x) - \phi(x) \int_{m_B}^x dy \phi(y)\tilde{\phi}^{-\gamma}(y) - \phi(x)\tilde{\phi}^{-\gamma}(x) \int_{m_B}^x dy \phi(y) \right] \right. \\
&\quad + \tilde{\phi}^{-\gamma}(m_B) \int_{m_B}^1 dx \left[(1 - m_B)\phi^2(x) + \phi(x) \int_{m_B}^1 dy \phi(y) \right] + 2\tilde{\phi}^{2-\gamma}(m_B) + \int_{m_B}^1 dx \left[(\tilde{\phi}^{-\gamma}(x) - \tilde{\phi}^{-\gamma}(m_B))(2\phi(x)\phi(\tilde{m}_B) - \tilde{\phi}^2(m_B)) \right. \\
&\quad \left. \left. - (\tilde{\phi}^{-\gamma}(x) + \tilde{\phi}^{-\gamma}(m_B))\phi^2(x) \right] \right\} \frac{1}{A} s(s+1) m_B^s \frac{1}{3} \left(1 - \frac{m}{m_B} \right)^3 + O((1 - m/m_B)^4). \tag{39}
\end{aligned}$$

Using the form of $q(x)$ defined in (30) we obtain, for $m_B < x_c$

$$\begin{aligned}
H_1(m_B, m) &= \frac{1}{48} As(s+1) 2m_B^{s+1} \left(1 - \frac{m}{m_B} \right)^4 \\
&\quad + O\left(\left(1 - \frac{m}{m_B} \right)^5 \right). \tag{34}
\end{aligned}$$

For the interaction terms, similarly

see equations (35) above

where we have used the auxiliary quantities

see equation (36) above.

In the last term we will write only the lowest order (in $(m_B - m)$) terms.

see equation (37) above.

Substituting the functions $q(x)$ and $r(x)$ we get the following results

see equations (38,39) above

where

$$\begin{aligned}\phi(x) &= r(x) - r(m_B) = -\frac{(s+1)^2}{As(s+2)} \left(\frac{1}{m_B^{s+2}} - \frac{1}{x^{s+2}} \right) \\ \tilde{\phi}(x) &= \tilde{r} - r(x) = -\frac{(s+1)^2}{As(s+2)} \\ &\quad \times \left(\frac{1}{x^{s+2}} - \frac{1}{x_c^{s+2}} + \frac{s+2}{s+1} \frac{1}{x_c^{s+1}} \right).\end{aligned}\quad (40)$$

4 Summing over permutations

Having computed all the matrix elements, we could proceed to computation of the free energy (22). For this end we first need to know the measure $d\mu_{\Pi}$ in the space of permutations. Starting with the discrete p -adic formulation of the replica symmetry breaking [20], where the size of the i th block is $m_i = p^i$ and $p^K = n$, we see that there are

$$\Delta P(m_B, m_i) = \frac{n(m_i - m_{i-1})}{2m_B^2} \quad (41)$$

permutations which exchange blocks of the size m_B at the co-distance m_i . The continuum limit corresponds to taking $p \rightarrow 1^-$, while $K \rightarrow \infty$ and $n = p^K \rightarrow 0$. Note that the usual replica limit $n \rightarrow 0$ and the limit which introduces the continuous replica symmetry breaking, $p \rightarrow 1^-$ should be taken independently, as in the spin-glass theory [1]. Thus, the measure in the set $S_n^{(2)}$ of the permutations in question is

$$d\mu^{(2)}(m_B, m) = -\frac{ndm_B dm}{2 \ln pm_B^3} \quad (42)$$

and it is immediately seen to diverge for $p \rightarrow 1^-$.

The meaning of the divergence clarifies itself once we take into account the rest of the set S_n^1 of all possible non-identical permutations. In the limit $n \rightarrow 0$ $|S_n^1| = n! - 1 = 0$ and the infinite number of permutations $\in S_n^{(2)}$ should be canceled by the contribution of other permutations. Actually, if we find *e.g.* the measure corresponding to the space $S_n^{(3)}$ of all cyclic exchanges of three blocks of replicas, we see that their number is infinite again, but with opposite sign than the number of permutations in $S_n^{(2)}$. It is that cancellation of infinities that makes finally the total number of non-identical permutations to be zero.

It would be very difficult task to perform the same calculations as those presented above for all the variety of possible permutations and to show the cancelation of the divergences explicitly. Instead, we will approximate the integral over all permutations with the exact measure by the much more simple integral over the set $S_n^{(2)}$ only, but taking a “renormalized” measure $d\bar{\mu}^{(2)}(m_B, m)$ instead of the true $d\mu^{(2)}(m_B, m)$. The interpretation we attribute to such a replacement is that we pick out the permutation Π with given m_B and m and suppose that the other permutations $\bar{\Pi}$, exchanging not only two, but

three, four *etc.* blocks of replicas are close to Π in the sense that corresponding T_{Π} and the matrix elements of the Hamiltonian are approximately equal, but when summing over all $\bar{\Pi}$'s, the divergences cancel, yielding the average measure $d\bar{\mu}^{(2)}(m_B, m)$ which is free of the divergence at $p \rightarrow 1$. Clearly, the above supposition is somewhat unfounded and its mathematical formulation is worth of further study. Nevertheless we consider the hypothesis to be plausible and useful at least for the purpose of establishing the power of the finite size corrections, which is what we are trying here. Thus, we will rely on it in the following.

The question then arises, how to choose the approximate measure $d\bar{\mu}^{(2)}(m_B, m)$. The measure has to meet the condition

$$\int_{m \in (0, m_B)} d\bar{\mu}^{(2)}(m_B, m) = d\mu^{(1)}(m_B) \quad (43)$$

where $d\mu^{(1)}(m_B)$ is the number of all permutations of block of the size between m_B and $m_B + dm_B$ with the restriction that they cannot be expressed as a permutation with another (smaller) block size. This restriction avoids multiple counting of the permutations. Clearly, the sum over the block sizes from 1 up to given m_B should give $(n/m_B)!$, so

$$\begin{aligned}d\mu^{(1)}(m_B) &= \frac{d}{dm_B} \Gamma \left(\frac{n}{m_B} + 1 \right) dm_B \\ &\stackrel{n \rightarrow 0}{=} -\Gamma'(1) \frac{n}{m_B^2} dm_B.\end{aligned}\quad (44)$$

Similarly, the following relation holds for the original measure $\mu^{(2)}$

$$\int_{m \in (0, m_B)} d\mu^{(2)}(m_B, m) = -\frac{n}{2 \ln pm_B^2} dm_B. \quad (45)$$

Hence, *per analogiam* we infer that

$$d\bar{\mu}^{(2)}(m_B, m) = -\Gamma'(1) \frac{n}{m_B^3} dmdm_B. \quad (46)$$

The same result comes out also following another consideration. Dividing all the permutations into disjunct groups according their characteristics m_B (by the same means as we did when we were deriving $\mu^{(1)}$) and within a selected group taking only such permutations which exchange two blocks, we obtain exactly the same result. This approach is based on the tacit assumption, that the pair exchanges dominate all quantities of interest. Actually, when we perform the calculation of $T_{\Pi} = T(m_B, m_1, m_2)$ in the case of a cyclic exchange of three blocks (the permutation is characterized by two co-distances $m_1 < m_2$) we find that $T(m_B, m_1, m_2) - T(m_B, m_1) \sim (1 - (m_2/m_B))^3 > 0$ and for large D the pair exchanges actually dominate.

The sentence just stated is slightly incorrect in that it holds only for $n > 1$. In the replica limit $n \rightarrow 0$ the inequality reverses and one would naively expect that, on the contrary, the pair exchanges are negligible. But the same consideration as the one which has clarified

the wrong sign in the exponential $\exp(-DT_{II}/2)$ applies here and the previous statement about the dominating permutations is restored.

Finally, we obtain for the free energy

$$F = H_0 - \Gamma'(1) \int \frac{dm dm_B}{m_B^3} H(m_B, m) \exp\left(-\frac{D}{2} T(m_B, m)\right) \quad (47)$$

where $H = H_1 + H_2 + H_3$. Note that the kinetic term, H_1 is of higher order in $(m_B - m)$ than the interaction one, $H_2 + H_3$, so the kinetic term can be omitted completely in the computation of the finite-dimensionality corrections.

Setting $\bar{m} = 1 - m/m_B$ we can see that the free energy has the form

$$F - H_0 = \int dm_B C(m_B) \int_0^1 d\bar{m} \bar{m}^3 e^{-\frac{D}{2}(-\frac{1}{3})(s+1)^2 \bar{m}^3} \quad (48)$$

and the integral over \bar{m} can be easily computed, yielding

$$\int_0^1 d\bar{m} \bar{m}^3 e^{-\frac{D}{2}(-\frac{1}{3})(s+1)^2 \bar{m}^3} \simeq -3^{5/3} 2^{4/3} \Gamma\left(\frac{4}{3}\right) (s+1)^{-8/3} D^{-4/3}, D \rightarrow \infty. \quad (49)$$

The main inference from this result is the D -dependence of the correction to the free energy, which is

$$F - H_0 \sim D^{-4/3}. \quad (50)$$

Of course, the full solution should continue with minimizing the free energy with respect of the matrix Q^{-1} , but the non-integer power of D in the corrections to free energy and propagator must persist. The open question and the most interesting one is, however, what are the finite- D corrections to the exponent s , which is the quantity which governs the long-distance behavior of a directed polymer. In answering this question, we cannot avoid the explicit computation of the stationary point of the free energy. In principle it is straightforward: we take the formulae just obtained and let the first derivatives with respect to the variational parameters s, A, x_c be zero. This work is in progress and we will refer the reader to further publication.

5 Conclusions

We have seen that infinitesimal permutations can be identified in the case of continuous breaking of the replica symmetry. The sum over these infinitesimal permutations is not a simple task: if one proceeds in a naive way the measure over a class of infinitesimal permutations is formally infinite and this infinity is related to having limited

our attention on a restricted class of permutations. The corresponding double counting problem gives rise to this divergence. We tentatively propose to cure this divergence by introducing an effective measure in order to take care of these double counting problems.

If we follow this tentative approach we can compute the shift in the ground state energy due to the presence of the tunneling and we find that in a given range of the parameters there are corrections which go to zero like $D^{-4/3}$. The effects of these corrections on the critical exponents have not been studied yet. However, we think that the tunneling among different states plays a crucial role in the final theory. It would be extremely interesting to compare these results with those coming from a standard 1/D expansion to see if these effects are correctly taken account in the perturbative 1/D expansion or they have a more non perturbative nature, as this study may suggest.

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